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GENERALIZATION OF THE METHOD OF FUNCTIONALLY INVARIANT SOLUTIONS FOR DYNAMIC PROBLEMS OF THE PLANE THEORY OF ELASTICITY OF ANISOTROPIC MEDIA†

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A new approach for constructing functionally invariant solutions for dynamic problems of the plane theory of elasticity of anisotropic media is proposed. Solutions of the equations of motion in displacements and potentials, which express plane waves and waves from a point source, and also complex solutions of a general type are obtained and investigated. The problem of the reflection of plane waves from the boundary of a half-space is solved for comparison with earlier results [1]. The solutions obtained agree with the physical meaning of the problems and with the solutions for isotropic media.

Solutions of the equations of motion in displacements and potentials respectively for anisotropic media with four and three constants of elasticity with certain limitations on the other constants were obtained in [1, 2] by the Smirnov and Sobolev method. Since a procedural inaccuracy was tolerated in these papers when constructing these solutions, the solutions obtained disagree somewhat with the physical meaning of the problem. In this paper we extend the investigation of these problems by the same method for anisotropic media with four constants of elasticity with no limitations on the constants of elasticity.

The Smirnov and Sobolev method of functionally invariant solutions [3–5], based on the idea of using the theory of functions of a complex variable to solve the wave equations, has become widely used to solve a number of important problems related to the propagation of transient elastic waves in isotropic media. It was pointed out in [6] that, after the basic results in the dynamics of elastic media obtained by Stokes, Rayleigh, Lamb and Love, the most important investigations were made by Smirnov and Sobolev, who developed a new logically faultless and mathematically rigorous method.

Wave processes in anisotropic media are complex and diverse, and basically depend on the ratios of the constants of elasticity and of the directions of propagation of the waves [7–14]. Unlike isotropic media, the equations of motion of anisotropic media do not reduce to wave equations, and well-known methods from the dynamics of isotropic media [6] have become widely used to integrate them. The effectiveness of these methods depends on the type of problems and class of anisotropic media.

1. SOLUTIONS OF THE WAVE EQUATIONS FOR AN ISOTROPIC MEDIUM

The functionally invariant solutions of the wave equations in the case of plane wave can be represented by the expressions [4]

$$\begin{aligned} &u_1 = \theta w_1(\Omega_1), \quad v_1 = \xi_1 w_1(\Omega_1) \\ &u_2 = \xi_2 w_2(\Omega_2), \quad v_2 = -\theta w_2(\Omega_2) \\ &\Omega_k = t + \theta x + \xi_k y, \quad k = 1, 2 \\ &\xi_1 = (1/a_0 - \theta^2)^{1/2}, \quad \xi_2 = (1/d_0 - \theta^2)^{1/2} \end{aligned} \tag{1.1}$$

where a_0 and d_0 are the squares of the phase velocities of longitudinal ($k = 1$) and transverse ($k = 2$) waves.

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Functions (1.1) in the sections

$$0 \leqslant \theta \leqslant a_0^{-1/2}, \quad 0 \leqslant \theta \leqslant d_0^{-1/2} \tag{1.2}$$

describe real longitudinal and transverse plane waves, propagating in directions defined by the inequalities

$$0 \leqslant \alpha_1 \leqslant \pi/2, \; 0 \leqslant \alpha_2 \leqslant \pi/2 \tag{1.3}$$

where α_1 and α_2 are the angles which the normals to the wave fronts make with the y axis. On the boundaries of sections (1.2), solutions (1.1) take the values:
when $\theta = 0$

$$u_1 = 0, \;\; v_1 = a_0^{-1/2} w_1(t + a_0^{-1/2} y) \tag{1.4}$$
$$u_2 = d_0^{-1/2} w_2(t + d_0^{-1/2} y), \;\; v_2 = 0$$

when $\theta = a_0^{-1/2}$ and $\theta = d_0^{-1/2}$

$$u_1 = a_0^{-1/2} w_1(t + a_0^{-1/2} x), \quad v_1 = 0 \tag{1.5}$$
$$u_2 = 0, \quad v_2 = -d_0^{-1/2} w_2(t + d_0^{-1/2} x)$$

Hence, solutions (1.1), defined in sections (1.2), describe the propagation of longitudinal and transverse plane waves in directions characterized by the continuously increasing angles α_1 and α_2 in the intervals (1.3).

2. SOLUTION OF THE EQUATIONS OF MOTION FOR AN ANISOTROPIC MEDIUM

The equations of motion for anisotropic media with four constants of elasticity have the form [1]

$$au_{xx} + du_{yy} + cv_{xy} = u_{tt}$$
$$dv_{xx} + bv_{yy} + cu_{xy} = v_{tt} \tag{2.1}$$

The ratios of the constants of elasticity to the density of the media

$$a = c_{11}/\rho, \quad b = c_{22}/\rho, \quad d = c_{66}/\rho, \quad c = (c_{66} + c_{12})/\rho \tag{2.2}$$

satisfy the necessary and sufficient conditions for the form of the elastic energy to be positive-definite

$$a > d, \;\; b > d, \;\; d > 0, \;\; ab - (c-d)^2 > 0 \tag{2.3}$$

which are satisfied for all practical media of the anisotropy class considered and are the necessary and sufficient conditions for elastic waves to propagate in any direction.

We will express the solution of system of equations (2.1) by the functions

$$u = U(\Omega), \quad v = V(\Omega) \tag{2.4}$$

where the function Ω is defined in implicit form by the equation

$$\delta \equiv l(\Omega)t + m(\Omega)x + n(\Omega)y + K(\Omega) = 0 \tag{2.5}$$

We will assume that U and V are continuous twice-differentiable functions if all the coefficients of the variable quantities in them are real. If some of these in any region are complex quantities, then U and V are analytical functions in this region. By determining the derivatives of function (2.4) using well-known formulae for differentiating complex and implicit functions [4] and substituting their values into ⁻ᵗem of equations (2.1), we obtain the conditions

$$(am^2+dn^2-l^2)U'(\Omega)+cmnV'(\Omega)=0$$
$$cmnU'(\Omega)+(dm^2+bn^2-l^2)V'(\Omega)=0 \tag{2.6}$$

which establish the relation between functions (2.4).

We will assume the determinant of system of equations (2.6) to be equal to zero

$$\Delta=(am^2+dn^2-l^2)(dm^2+bn^2-l^2)-c^2m^2n^2=0 \tag{2.7}$$

Functions (2.4) express the solution of the system of equations of motion (2.1) if the argument Ω is defined by Eq. (2.5) with coefficients which are subject to Eq. (2.7), while the functions themselves satisfy conditions (2.6).

Taking $l(\Omega)=1, m(\Omega)=\theta, n(\Omega)=\lambda$ and $K(\Omega)=-\Omega$ in expressions (2.5)–(2.7), we obtain the simplest solutions of system of equations (2.1), which represent plane waves

$$u_k=U_k(\Omega_k),\quad v_k=V_k(\Omega_k) \tag{2.8}$$
$$\Omega_k=t+\theta x+\lambda_k y,\quad k=1,2$$

where λ_k are the roots of Eq. (2.7), which are functions of the variable θ

$$\lambda_k=\{H+(-1)^k[H^2-(a/b)(1/a-\theta^2)(1/d-\theta^2)]^{1/2}\}^{1/2} \tag{2.9}$$
$$H=[(b+d)-(ab+d^2-c^2)\theta^2]/(2bd)$$

The functions $U_k(\Omega_k)$ and $V_k(\Omega_k)$ correspond to the roots λ_k and, according to expressions (2.6), satisfy the conditions

$$p_kU_k'(\Omega_k)+c\theta\lambda_kV_k'(\Omega_k)=0,\quad c\theta\lambda_kU_k'(\Omega_k)+r_kV_k'(\Omega_k)=0 \tag{2.10}$$
$$p_k=a\theta^2+d\lambda_k^2-1,\quad r_k=d\theta^2+b\lambda_k^2-1,$$
$$p_kr_k=c^2\theta^2\lambda_k^2$$

The question arises of which of conditions (2.10) needs to be chosen to construct the solution. The first condition was chosen previously in [1].

The first condition of (2.10) leads to a solution of the equations of motion of the form

$$u_k=+c\theta\lambda_kw_k(\Omega_k),\quad v_k=-p_kw_k(\Omega_k) \tag{2.11}$$

while the second condition of (2.10) leads to a solution of the form

$$u_k=r_kw_k(\Omega_k),\quad v_k=-c\theta\lambda_kw_k(\Omega_k) \tag{2.12}$$

Solutions (2.11) take the following values:
when $\theta=0$

$$u_1=0,\quad v_1=\frac{b-d}{b}w_1(t+b^{-1/2}y);\quad u_2=0,\quad v_2=0 \tag{2.13}$$

when $\theta=a^{-1/2}$ $(k=1)$ and $\theta=d^{-1/2}$ $(k=2)$

$$u_1=0,\quad v_1=0,\quad u_2=0,\quad v_2=-\frac{a-d}{d}w_2(t+d^{-1/2}x) \tag{2.14}$$

Solutions (2.12) take the following values:
when $\theta_1=0$

$$u_1=0,\quad v_1=0,\quad u_2=\frac{b-d}{d}w_2(t+d^{-1/2}y),\quad v_2=0 \tag{2.15}$$

when $\theta=a^{-1/2}(k=1)$ and $\theta=d^{-1/2}(k=2)$

$$u_1 = -\frac{a-d}{a} w_1(t + a^{-1/2}x), \quad v_1 = 0, \quad u_2 = 0, \quad v_2 = 0 \tag{2.16}$$

If follows from (2.13)–(2.16) that solutions (2.11) and (2.12) at the boundaries of the sections $(0, a^{-1/2})$ and $(0, d^{-1/2})$ have different values and do not agree with expressions (1.4) and (1.5) of the similar solutions (1.1) for an isotropic medium. In this case, each of the solutions (2.11) and (2.12), obtained from one of the two conditions (2.10), defines the propagation of one type of wave in the directions of the axes of elastic symmetry of the medium, namely, quasi-longitudinal or quasi-transverse waves, which does not agree with the physical meaning of the problem. This can be explained by the fact that the ratios of the components of the displacement vectors $u_k/v_k = -(c\theta\lambda_k)/p_k$ of the quasi-transverse and quasi-longitudinal waves of solutions (2.11) when $\theta = 0$ and $\theta = a^{-1/2}$, and the ratios $u_k/v_k = -r_k/(c\theta\lambda_k)$ of the quasi-longitudinal and quasi-transverse waves of solutions of (2.12) when $\theta = 0$ and $\theta = d^{-1/2}$ have an indeterminacy of the form 0/0.

It follows from (2.13)–(2.16) that, summing solutions (2.11) and (2.12), we obtain solutions of system of equations (2.1), which define the propagation of quasi-longitudinal and quasi-transverse waves for $\theta = 0$, $\theta = a^{-1/2}$ $(k = 1)$ and $\theta = d^{-1/2}$ $(k = 2)$. These solutions can be obtained directly using generalized conditions, which establish the relation between functions (2.8).

Summing the left-hand sides of expressions (2.10), we obtain a generalized condition of the form

$$(p_k + c\theta\lambda_k)U_k'(\Omega_k) + (r_k + c\theta\lambda_k)V_k'(\Omega_k) = 0 \tag{2.17}$$

By (2.17) the solutions of the equations of motion (2.1), which describe plane waves, have the form

$$u_k = (r_k + c\theta\lambda_k)w_k(\Omega_k), \quad v_k = -(p_k + c\theta\lambda_k)w_k(\Omega_k) \tag{2.18}$$
$$\Omega_k = t + \theta x + \lambda_k y, \quad k = 1, 2$$

where w_k are arbitrary continuous twice-differentiable functions, if the coefficients of the variable quantities are real. If some of these coefficients in any region of space x, y, t are complex quantities, then w_k are analytical functions in this region. The solutions obtained satisfy condition (2.17), each of conditions (2.10) and system of equations (2.1), as can easily be shown by direct substitution.

Solutions (2.18), for values of θ corresponding to the propagation of waves in the direction of the y and x axes, take the following values:
when $\theta = 0$

$$u_1 = 0, \quad v_1 = \frac{b-d}{b} w_1(t + b^{-1/2}y)$$
$$u_2 = \frac{b-d}{d} w_2(t + d^{-1/2}y), \quad v_2 = 0 \tag{2.19}$$

when $\theta = a^{-1/2}$ $(k = 1)$ and $\theta = d^{-1/2}$ $(k = 2)$

$$u_1 = -\frac{a-d}{a} w_1(t + a^{-1/2}x), \quad v_1 = 0$$
$$u_2 = 0, \quad v_2 = -\frac{a-d}{d} w_2(t + d^{-1/2}x) \tag{2.20}$$

similar to the values (1.4) and (1.5) for an isotropic medium.

For values of the constants of elasticity $b = a$ and $c = a-d$, corresponding to an isotropic medium, (2.18) reduce, apart from the constant factors A and B, to the solutions (1.4) of the wave equations

$$u_1 = A\theta w_1(\Omega_1), \quad v_1 = A\xi_1 w_1(\Omega_1) \tag{2.21}$$
$$u_2 = B\xi_2 w_2(\Omega_2), \quad v_2 = -B\theta w_2(\Omega_2)$$
$$\Omega_k = t + \theta x + \xi_k y, \quad k = 1, 2$$

The functions λ_1 and λ_2, represented by expressions (2.9), are branches of the algebraic function λ, uniquely defined on the Riemann surface, the form of which depends on the ratios of the constants of elasticity.

For the condition [13]

$$(a-d)b-c^2>0 \tag{2.22}$$

the branching points for the outer radicals (2.9) are the points $\theta_1 = \pm a^{-1/2}$ when $k = 1$ and the points $\theta_2 = \pm d^{-1/2}$ when $k = 2$, and for the inner radicals the points θ_i°, which, depending on the ratios of the constants of elasticity, may be complex, imaginary or real. The Riemann surface consists of the planes θ_1 and θ_2 with cuts $A = (-a^{-1/2}, +a^{-1/2})$ and $D = (-d^{-1/2}, +d^{-1/2})$, joined criss-cross along the cuts which join the branching points θ_i°. In Fig. 1 we show the Riemann surface for the case when the branching points θ_i° are pairwise complex conjugate.

On the sides of the cuts A of the plane θ_1 and D of the plane θ_2, the function λ_1 and λ_2 have real values, and the functions (2.18) express real plane waves: quasi-longitudinal for $k = 1$ and quasi-transverse for $k = 2$, which propagate with normal velocities and directions of motion, defined by the formulae [12]

$$b_k = (\theta^2+\lambda_k^2)^{-1/2}, \quad \operatorname{tg}\alpha_k = \theta/\lambda_k \tag{2.23}$$

where α_k are the angles which the normals to the wave fronts make with the y axis. By fixing the functions λ_1 and λ_2 in the θ_1 and θ_2 planes of the Riemann surface so that they are positive when $\theta = i\beta$, where β is a fairly small positive quantity, it is sufficient to investigate the real solutions (2.18) on the upper sides of the cuts A of the θ_1 plane and D of the θ_2 plane for positive values of θ, since the anisotropic medium is symmetrical about the x and y axes.

In the sections

$$0 \leqslant \theta \leqslant a^{-1/2}, \quad 0 \leqslant \theta \leqslant d^{-1/2} \tag{2.24}$$

of the upper sides of the cuts of the θ_1 and θ_2 planes, in accordance with formulae (2.23), the functions (2.18) express quasi-longitudinal ($k = 1$) and quasi-transverse ($k = 2$) waves, which propagate with continuously increasing angles α_1 and α_2 in the intervals

$$0 \leqslant \alpha_1 \leqslant \pi/2, \quad 0 \leqslant \alpha_2 \leqslant \pi/2 \tag{2.25}$$

and with continuously varying normal velocities with values on the boundaries of the sections

$$b_1(0) = b^{1/2}, \quad b_1(a^{-1/2}) = a^{-1/2}, \quad b_2(0) = b_2(d^{-1/2}) = d^{1/2} \tag{2.26}$$

When the condition

$$(a-d)b-c^2<0 \tag{2.27}$$

is satisfied, which was not considered in [1], the Riemann surface has a different form (Fig. 2). The outer radical of the function λ_1 has four branching points: $\theta_1 = \pm a^{-1/2}$ and $\theta_2 = \pm d^{-1/2}$, and the outer

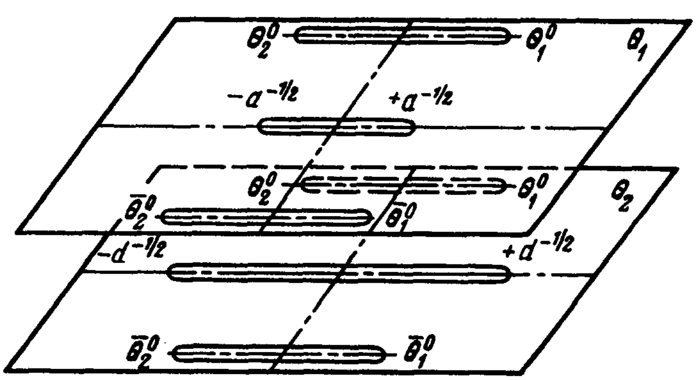

Fig. 1

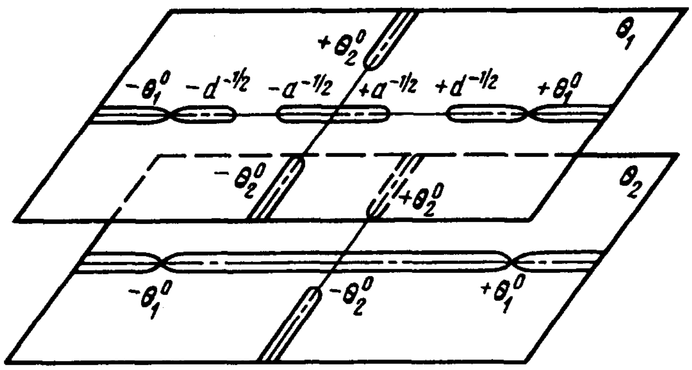

Fig. 2

radical of the function λ_2 has no branching points [12]. From the four branching points for the inner radical of the function λ_1 and λ_2 we have: two real $\pm\theta_1^\circ$ and two imaginary $\pm\theta_2^\circ$, where $\theta_1^\circ > d^{-1/2}$. The function λ_1 is unique in the θ_1 plane with cuts A, $(\pm d^{-1/2}, \pm\theta_1^\circ)$ and $(\pm\theta_1^\circ, \pm\infty)$ along the real axis and with cuts $(\pm\theta_2^\circ, \pm i\infty)$ along the imaginary axis. The function λ_2 is unique in the θ_2 plane with cuts $C = (-\theta_1^\circ, +\theta_1^\circ)$ along the real axis and with cuts $(\pm\theta_2^\circ, \pm i\infty)$ along the imaginary axis. The Riemann surface consists of the θ_1 and θ_2 planes, joined criss-cross along the edges of the cuts $(\pm\theta_1^\circ, \pm\infty)$ and $(\pm\theta_2^\circ, \pm i\infty)$.

On the edges of the cuts A and $(\pm d^{-1/2}, \pm\theta_1^\circ)$ of the θ_1 plane and C of the θ_2 plane the functions λ_1 and λ_2 have real values, and the functions (2.18) represent real waves.

In the sections

$$0 \leqslant \theta \leqslant a^{-1/2}, \quad 0 \leqslant \theta \leqslant \theta_1^\circ \tag{2.28}$$

of the upper edges of the cuts of the θ_1 and θ_2 planes, the functions (2.18), in accordance with formulae (2.23), represent quasi-longitudinal ($k = 1$) and quasi-transverse ($k = 2$) waves, propagating with continuously increasing angles α_1 and α_2 in the intervals

$$0 \leqslant \alpha_1 \leqslant \pi/2, \quad 0 \leqslant \alpha_2 \leqslant \alpha_2^\circ \ (\alpha_2^\circ < \pi/2) \tag{2.29}$$

and with continuously varying normal velocities with values at the boundaries of the sections [12]

$$b_1(0) = b^{1/2}, \quad b_1(a^{-1/2}) = a^{1/2}, \quad b_2(0) = d^{1/2}, \quad b_2(\theta_1^\circ) < d^{1/2} \tag{2.30}$$

In the section $(+a^{-1/2}, +d^{-1/2})$ of the θ_1 plane the function λ_1 takes imaginary values $\lambda_1 = -i\lambda_1^*$, and the functions (2.18) for $k = 1$ describe complex quasi-longitudinal waves

$$u_1 = (r_1 - c\theta \cdot i\lambda_1^*)w_1(\Omega_1^*), \quad v_1 = -(p_1 - c\theta \cdot i\lambda_1^*)w_1(\Omega_1^*)$$
$$\Omega_1^* = t + \theta x - i\lambda_1^* y$$

At the upper edge of the cut $B = (+d^{-1/2}, +\theta_2^\circ)$ of the θ_1 plane the function λ_1 takes real values $-\lambda_1$. In the sections B of the upper edges of the cuts B of the θ_1 plane and C of the θ_2 plane the functions (2.18) have the form

$$\begin{aligned} u_1 &= (r_1 - c\theta\lambda_1)w_1(\Omega_1^-), \quad v_1 = -(p_1 - c\theta\lambda_1)w_1(\Omega_1^-) \\ u_2 &= (r_2 + c\theta\lambda_2)w_2(\Omega_2^+), \quad v_2 = -(p_2 + c\theta\lambda_2)w_2(\Omega_2^+) \\ \Omega_k^\pm &= t + \theta x \pm \lambda_k y \end{aligned} \tag{2.31}$$

and describe real waves.

When going clockwise round the branching point θ_1° from the upper edges of the cuts C of the θ_2 plane and B of the θ_1 plane, on the lower edges of the cuts B of the θ_1 and C of the θ_2 plane the functions λ_2 and λ_1 take the values λ_1 and λ_2, respectively. On the lower edges of the cuts B and C the functions (2.31) take the form

$$u_1 = (r_1 + c\theta\lambda_1)w_1(\Omega_1^+), \quad v_1 = -(p_1 + c\theta\lambda_1)w_1(\Omega_1^+)$$
$$u_2 = (r_2 - c\theta\lambda_2)w_2(\Omega_2^-), \quad v_2 = -(p_2 - c\theta\lambda_2)w_2(\Omega_2^-) \tag{2.32}$$

The function u_2 and v_2 in expressions (2.31), representing quasi-transverse waves, and the functions u_1 and v_1 in expressions (2.32) at the branching point θ_1° have the same values. Consequently, the functions (2.18) in the sections B of the upper and lower edges of the cuts of the θ_1 and θ_2 planes of the Riemann surface (Fig. 2) only describe quasi-transverse waves. This feature has a direct connection with the existence of acute-angled edges on the quasi-transverse wave fronts from a point source [13] when condition (2.27) is satisfied.

In the section

$$+d^{-1/2} \leq \theta \leq \theta_1^\circ \tag{2.33}$$

of the lower edge of the cut of the θ_1 plane the functions (2.32) when $k = 1$ describe quasi-transverse waves propagating in accordance with formulae (2.23), with continuously decreasing angles α_1 and normal velocities b_1 in the intervals

$$\pi/2 \geq \alpha_1 \geq \alpha_1(\theta_1^\circ) = \alpha_2(\theta_1^\circ), \quad d^{1/2} \geq b_1 \geq b_1(\theta_1^\circ) = b_2(\theta_1^\circ) \tag{2.34}$$

On the edges of the cuts $(+\theta_1^\circ, +\infty)$ of the θ_1 and θ_2 planes the functions λ_1 and λ_2 have complex values and the functions (2.18) describe complex waves.

Hence, solutions (2.18) of the equations of motion (2.1), describing plane waves, depending on conditions (2.22) and (2.27), for the constants of elasticity, are uniquely defined on the real axes of the θ_1 and θ_2 planes of the Riemann surfaces, shown in Figs. 1 and 2, and uniquely describe the propagation of waves in any directions.

We obtain complex solutions of a general type of the equations of motion (2.1) by taking $l(\Omega) = 1$, $n(\Omega) = \lambda$, in Eqs (2.5)–(2.7), and taking $\theta = \mathrm{m}(\Omega)$ as the new variable. In this case $K(\Omega)$ will be a function of the variable θ, while the roots λ_1 and λ_2 of Eq. (2.7), defined by expression (2.9) and representing branches of the algebraic function λ, are uniquely defined on the Riemann surface (Figs 1 and 2).

Solutions (2.8) of Eqs (2.1) take the form

$$u_k = U_k(\theta_k), \quad v_k = V_k(\theta_k) \tag{2.35}$$

while Eq. (3.5) takes the form

$$\delta_k = t + \theta_k x + \lambda_k y + K_k(\theta_k) = 0 \tag{2.36}$$

where K_k are the branches of a certain analytical function K.

The generalized condition, which establishes relations between functions (2.35), takes the form

$$U_k'(\theta_k)/(r_k + c\theta\lambda_k) = -V_k'(\theta_k)/(p_k + c\theta\lambda_k) = w_k(\theta_k) \tag{2.37}$$

By (2.37), the general real solution of Eqs (2.1) is expressed by the functions

$$u = \sum_{k=1}^{2} \operatorname{Re} \int^{\theta_k} (r_k + c\zeta\lambda_k)w_k(\zeta)d\zeta$$
$$v = -\sum_{k=1}^{2} \operatorname{Re} \int^{\theta_k} (p_k + c\zeta\lambda_k)w_k(\zeta)d\zeta \tag{2.38}$$

$$\lambda_k = \{H + (-1)^k[H^2 - (a/b)(1/a - \zeta^2)(1/d - \zeta^2)]^{1/2}\}^{1/2}$$
$$H = [(b+d) - (ab + d^2 - c^2)\zeta^2]/(2bd)$$
$$p_k = a\zeta^2 + d\lambda_k^2 - 1, \quad r_k = d\zeta^2 + b\lambda_k^2 - 1$$

Solution (2.38), depending on the ratios of the constants of elasticity (2.22) and (2.27), is uniquely defined on the Riemann surfaces shown in Figs 1 and 2. The correspondence between the points of the Riemann surface and the points of the xy plane is expressed by Eq. (2.36). The functions λ_1 and λ_2 have real values on the edges of the cuts A of the θ_1 plane and D of the θ_2 plane (Fig. 1) under conditions (2.22) and on the edges of the cuts A and $(\pm d^{-1/2}, \pm\theta_1^\circ)$ of the θ_1 plane and C of the θ_2 plane (Fig. 2) for condition (2.27). The wave fronts of the quasi-longitudinal and quasi-transverse waves (2.38) can be obtained as the envelope of the straight lines (2.36) for real values of θ_k and λ_k. The functions w_1 and w_2 are branches of the arbitrary analytical function w, and the functions are chosen so that the real parts of the functions w_1 and w_2 vanish on the edges of the cuts of the θ_1 and θ_2 planes, which have real values of the functions λ_1 and λ_2.

Uniform zero-dimensional solutions of Eqs (2.1), which express the propagation of elastic vibrations in an anisotropic medium from a point source of the instantaneous-pulse type at the origin of coordinates, can be obtained as a special case of the general solutions, if we put $K_k(\theta_k) = 0$ in Eqs (2.36). In this case Eq. (2.36) takes the form

$$1+\theta_k\zeta+\lambda_k\eta=0 \quad (\zeta=x/t, \quad \eta=y/t) \tag{2.39}$$

The uniform solutions are expressed by the functions (2.38) and differ in form from the solutions obtained in [1].

3. SOLUTION OF THE PROBLEM OF THE REFLECTION OF PLANE WAVES AT THE BOUNDARY OF AN INISOTROPIC HALF-SPACE

The problem of the reflection of plane waves at the boundary of an anisotropic half-space with four constants of elasticity was solved in [1] using expressions (2.11) for plane waves, obtained by employing the first condition of (2.10) for the components of the displacement vectors.

For comparison with these solutions, we will consider the solution of this problem by expressing the quasi-longitudinal and quasi-transverse waves by the functions (2.18), obtained using generalized condition (2.17).

Suppose a quasi-longitudinal wave

$$u_1=U_1(t-\theta x+\lambda_1 y), \quad v_1=V_1(t-\theta x+\lambda_1 y) \tag{3.1}$$

is incident on a stress-free boundary $y = 0$ from an anisotropic half-space $y > 0$.

By relations (2.18), we can express the incident quasi-longitudinal wave and the reflected quasi-longitudinal and quasi-transverse waves by the functions

$$\begin{gathered}
u_1=\tilde{\varphi}_1^\circ w_1(\Omega_1^+), \quad v_1=-\tilde{\psi}_1^\circ w_1(\Omega_1^+) \\
u_{11}=\varphi_1^\circ A_1 w_1(\Omega_1^-), \quad v_{11}=-\psi_1^\circ A_1 w_1(\Omega_1^-) \\
u_{21}=\varphi_2^\circ B_1 w_1(\Omega_2^-), \quad v_{21}=-\psi_2^\circ B_1 w_1(\Omega_2^-) \\
\Omega_k^\pm=t-\theta x\pm\lambda_k y \quad (k=1,2) \\
\varphi_k^\circ=r_k+c\theta\lambda_k, \quad \psi_k^\circ=p_k+c\theta\lambda_k \\
\tilde{\varphi}_k^\circ=r_k-c\theta\lambda_k, \quad \tilde{\psi}_k^\circ=p_k-c\theta\lambda_k
\end{gathered} \tag{3.2}$$

Substituting the values $U = u_1 + u_{11} + u_{21}$ and $V = v_1 + v_{11} + v_{21}$ into the boundary conditions [1]

$$(c-d)\frac{\partial U}{\partial x}+b\frac{\partial V}{\partial y}=0, \quad \frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}=0 \quad (y=0) \tag{3.3}$$

we obtain the system of equations

$$\begin{gathered}
[b\lambda_1\psi_1^\circ-(c-d)\theta\varphi_1^\circ]A_1+[b\lambda_2\psi_2^\circ-(c-d)\theta\varphi_2^\circ]B_1=b\lambda_1\tilde{\psi}_1^\circ+(c-d)\theta\tilde{\varphi}_1^\circ \\
[\theta\psi_1^\circ-\lambda_1\varphi_1^\circ]A_1+[\theta\psi_2^\circ-\lambda_2\varphi_2^\circ]B_1=-\theta\tilde{\psi}_1^\circ-\lambda_1\tilde{\varphi}_1^\circ
\end{gathered} \tag{3.4}$$

Solving system (3.4) by the method of determinants, we obtain

$$A_1 = -(\tilde{\psi}_1^\circ / \psi_1^\circ)A_{11}, \quad B_1 = -(p_2\tilde{\psi}_1^\circ / p_1\psi_2^\circ)A_{21} \tag{3.5}$$

where

$$A_{11} = R^+ / R^-, \quad A_{21} = S(\lambda_1)/Q^- \tag{3.6}$$

Here

$$R^\pm = (\lambda_1 \pm \lambda_2)\left\{\{[ab-(c-d)^2]\theta^2 - b\}(1/d-\theta^2)^{1/2} \pm (ab)^{1/2}(1/a-\theta^2)^{1/2}\right\}$$

$$S(\lambda_k) = 2\lambda_k\{[ab-c(c-d)]\theta^2 + bd\lambda_k^2 - b\}[a\theta^2-(c-d)\lambda_k^2-1] \tag{3.7}$$

$$Q^- = c(a/b)^{1/2}(1/a-\theta^2)^{1/2}R^-$$

Unlike existing results [1] the factors $(\lambda_1 + \lambda_2)$, λ_1 and $(\lambda_1 - \lambda_2)$ are included in the quantities R^+, $S(\lambda_1)$ and R^-.

Substituting (3.5) into (3.2) and taking into account the relations

$$\varphi_k^\circ / \psi_k^\circ = c\theta\lambda_k / p_k, \quad \tilde{\psi}_k^\circ = -p_k\tilde{\varphi}_k^\circ / c\theta\lambda_k$$

which arise from relations (3.2) and (2.10), we obtain the solution of the problem of the reflection of quasi-longitudinal waves in the form

$$\begin{aligned} u_1 &= \tilde{\varphi}_1^\circ w_1(\Omega_1^+), \quad v_1 = -\tilde{\psi}_1^\circ w_1(\Omega_1^+) \\ u_{11} &= \tilde{\varphi}_1^\circ A_{11} w_1(\Omega_1^-), \quad v_{11} = \tilde{\psi}_1^\circ A_{11} w_1(\Omega_1^-) \\ u_{21} &= \tilde{\varphi}_1^\circ(\lambda_2/\lambda_1)A_{21}w_1(\Omega_2^-), \quad v_{21} = \tilde{\psi}_1^\circ(p_2/p_1)A_{21}w_1(\Omega_2^-) \end{aligned} \tag{3.8}$$

Taking into account the fact that the incident quasi-longitudinal wave satisfies condition (2.17), we can write solution (3.8) in the form

$$\begin{aligned} u_1 &= U_1(\Omega_1^+), \quad v_1 = V_1(\Omega_1^+) \\ u_{11} &= A_{11}U_1(\Omega_1^-), \quad v_{11} = -A_{11}V_1(\Omega_1^-) \\ u_{21} &= (\lambda_2/\lambda_1)A_{21}U_1(\Omega_2^-), \quad v_{21} = -(p_2/p_1)A_{21}V_1(\Omega_2^-) \end{aligned} \tag{3.9}$$

The solution of the problem of the reflection of quasi-transverse waves is constructed in the same way and has the form

$$\begin{aligned} u_2 &= \tilde{\varphi}_2^\circ w_2(\Omega_2^+), \quad v_2 = -\tilde{\psi}_2^\circ w_2(\Omega_2^+) \\ u_{12} &= -\tilde{\varphi}_2^\circ(\lambda_1/\lambda_2)A_{12}w_2(\Omega_1^-), \quad v_{12} = -\tilde{\psi}_2^\circ(p_1/p_2)A_{12}w_2(\Omega_1^-) \\ u_{22} &= -\tilde{\varphi}_2^\circ A_{22}w_2(\Omega_2^-), \quad v_{22} = -\tilde{\psi}_2^\circ A_{22}w_2(\Omega_2^-) \end{aligned} \tag{3.10}$$

where

$$A_{12} = S(\lambda_2)/Q^-, \quad A_{22} = R^+/R^- \tag{3.11}$$

and $S(\lambda_2)$, Q^- and $R^\pm$ are defined by (3.7). Unlike existing results [1], the factors $2\lambda_2$, $(\lambda_1 + \lambda_2)$ and $(\lambda_1 - \lambda_2)$ are included in the values of the constants $S(\lambda_2)$, R^+ and R^-.

Taking into account the fact that the incident quasi-transverse wave satisfies condition (2.17), solution (3.10) can be written in the form

$$
\begin{aligned}
&u_2 = U_2(\Omega_2^+),\quad v_2 = V_2(\Omega_2^+) \\
&u_{12} = -(\lambda_1/\lambda_2)A_{12}U_2(\Omega_1^-),\quad v_{12} = (p_1/p_2)A_{12}V_2(\Omega_1^-) \\
&u_{22} = -A_{22}U_2(\Omega_2^-),\quad v_{22} = A_{22}V_2(\Omega_2^-)
\end{aligned}
\tag{3.12}
$$

Solutions (3.8)–(3.10) and (3.12) are uniquely defined on the Riemann surfaces shown in Fig. 1 with condition (2.22), and Fig. 2 for condition (2.27) for the constants of elasticity. The case (2.27) was not considered previously in [1].

The solutions of the problems of the reflection of quasi-longitudinal and quasi-transverse waves of the form (3.8) and (3.10) differ from the similar solutions obtained previously in [1] by the presence of the factors $\bar{\varphi}_1^\circ$, $\bar{\psi}_1^\circ$ and $\bar{\varphi}_2^\circ$, $\bar{\psi}_2^\circ$. The coefficients A_{11}, A_{21} and A_{12}, A_{22} have the same values, since they are determined by the boundary conditions.

Quasi-longitudinal and quasi-transverse waves, when propagating in the directions of the axes of symmetry of the medium y (when $\theta = 0$) and x (when $\theta = a^{-1/2}$ and $\theta = -d^{-1/2}$) become purely longitudinal and purely transverse waves.

When the incident quasi-longitudinal and quasi-transverse waves travel in directions normal to the boundary of the medium and along the boundary, solutions (3.8) and (3.10) take the following values:

Solution (3.8) when $\theta = 0$

$$
\begin{aligned}
&v_1 = (b-d)b^{-1}w_1\left(t + b^{-1/2}y\right),\quad v_{11} = (b-d)b^{-1}w_1\left(t - b^{-1/2}y\right) \\
&u_1 = u_{11} = u_{21} = v_{21} = 0
\end{aligned}
\tag{3.13}
$$

when $\theta = a^{-1/2}$

$$
\begin{aligned}
&u_1 = -(a-d)a^{-1}w_1(t-\theta x),\quad u_{11} = (a-d)a^{-1}w_1(t-\theta x) \\
&v_1 = v_{11} = u_{21} = v_{21} = 0
\end{aligned}
\tag{3.14}
$$

solution (3.10) when $\theta = 0$

$$
\begin{aligned}
&u_2 = (b-d)d^{-1}w_2\left(t + d^{-1/2}y\right),\quad u_{22} = (b-d)d^{-1}w_2\left(t - d^{-1/2}y\right) \\
&v_2 = u_{12} = v_{12} = v_{22} = 0
\end{aligned}
\tag{3.15}
$$

when $\theta = d^{-1/2}$

$$
\begin{aligned}
&v_2 = (a-d)d^{-1}w_2\left(t - d^{-1/2}x\right),\quad v_{22} = -(a-d)d^{-1}w_2\left(t - d^{-1/2}x\right) \\
&u_2 = u_{22} = u_{12} = v_{12} = 0
\end{aligned}
\tag{3.16}
$$

The values (3.13)–(3.16) agree with the corresponding values in the similar problem for an isotropic half-space [15].

For values of the constants of elasticity $b = a$ and $c = a - d$, solutions (3.8) and (3.10) reduce to the solutions of the similar problem for an isotropic half-space [4].

We will compare the solutions of the problems of the reflection of quasi-longitudinal and quasi-transverse waves in the form (3.9) and (3.12) with the similar solutions obtained previously in [1]. Substituting the quantities (3.6) into solutions (3.9) we obtain solutions (43) and (44) from [1]. Taking into account the fact that in [1] $A = -A_{11}$ and $B = -A_{21}$, the solution of the problem of the reflection of quasi-longitudinal waves (43) and (44) reduces to the form (3.9). We can similarly show that the solution of the problem of the reflection of quasi-transverse waves obtained in [1] reduces to the form (3.12), and the intensity coefficients have the same values, since they are determined by the boundary conditions.

Hence, solutions (3.9) and (3.12) are identical in their external form with the similar solutions obtained in [1], with the sole difference that the relation between U_j and V_j in solutions (3.9) and (3.12) is established by the generalized condition (2.17), while in the solutions from [1] it is established by the first condition of (2.10).

Comparing solutions (3.9) and (3.8) and also (3.12) and (3.10), we obtain

$$U_j(\Omega_k^\pm) = \tilde{\varphi}_j^\circ w_j(\Omega_k^\pm), \quad V_j(\Omega_k^\pm) = -\tilde{\psi}_j^\circ w_j(\Omega_k^\pm), \quad j = 1,2 \tag{3.17}$$

It follows from (3.17), on the basis of the results of an analysis of solutions (2.18) of the equations of motion (2.1), that solutions (3.9) and (3.12) agree with the solutions of the similar problem for an isotropic medium.

In the solutions obtained in [1], the functions U_j and V_j have the following values:

$$U_j(\Omega_k^\pm) = c\theta\lambda_j w_j(\Omega_k^\pm), \quad V_j(\Omega_k^\pm) = p_j w_j(\Omega_k^\pm), \quad j = 1,2 \tag{3.18}$$

According to the results of an analysis of solutions (2.11), obtained on the basis of the first condition of (3.10), it follows from (3.18) that the solutions of the reflection problem [1], expressed in terms of the functions U_j and V_j, do not agree with the solutions of the similar problem for an isotropic medium when the incident waves in the directions of the axes of elastic symmetry of the medium are considered.

Note that, if in the solutions obtained in [1], we assume that the functions U_j and V_j satisfy the generalized condition (2.17), the functions U_j and V_j will take the values (3.17), while the solutions themselves will be identical with solutions (3.9) and (3.12), since the intensity coefficients in them have the same values. This can be shown by substituting the values (3.17) into the solutions obtained previously in [1]. As a result we have solutions (3.8) and (3.10), which are equivalent to solutions (3.9) and (3.12).

4. THE EQUATIONS OF MOTION IN POTENTIALS AND THEIR SOLUTION

An arbitrary vector field can be represented in the form of the sum of an irrotational field and a solenoidal field. Introducing the potentials of the irrotational and solenoidal perturbations using the formula [2]

$$u = \varphi_x + \psi_y, \quad v = \varphi_y - \psi_x \tag{4.1}$$

we obtain the equations of motion (2.1) in potentials

$$\left[a\varphi_{xx} + (d+c)\varphi_{yy} - \varphi_{tt}\right]_x + \left[(a-c)\psi_{xx} + d\psi_{yy} - \psi_{tt}\right]_y = 0 \tag{4.2}$$

$$\left[(d+c)\varphi_{xx} + b\varphi_{yy} - \varphi_{tt}\right]_y - \left[d\psi_{xx} + (b-c)\psi_{yy} - \psi_{tt}\right]_x = 0$$

We will express the solution of system of equations (4.2) by the functions

$$\varphi = \Phi(\Omega), \quad \psi = \Psi(\Omega) \tag{4.3}$$

where function Ω is defined in implicit form by Eq. (2.5). We mean by Φ and Ψ continuous triply differentiable functions, if the coefficients of the variables in them are real. If some of these coefficients in some region of space x, y, t are complex quantities, then Φ and Ψ will be analytical functions in this region.

Expressing the derivatives of function (4.3) using the ordinary rules of differentiation of complex and implicit functions [2] and substituting their values into Eqs (4.2), we obtain the conditions

$$m\left[am^2 + (d+c)n^2 - l^2\right]\Phi' + n\left[(a-c)m^2 + dn^2 - l^2\right]\Psi' = 0$$

$$n\left[(d+c)m^2 + bn^2 - l^2\right]\Phi' - m\left[dm^2 + (b-c)n^2 - l^2\right]\Psi' = 0 \tag{4.4}$$

which establish the relation between functions (4.3). If functions (4.3) satisfy conditions (4.4), they will also satisfy system of equations (4.2).

System of linear equations (4.4) has non-zero solutions if its determinant is equal to zero

$$m^2\left[am^2+(d+c)n^2-l^2\right]\left[dm^2+(b-c)n^2-l^2\right]+$$
$$+n^2\left[(a-c)m^2+dn^2-l^2\right]\left[(d+c)m^2+bn^2-l^2\right]=0 \qquad (4.5)$$

Equation (4.5) establishes a relation between the functions $l(\Omega)$, $m(\Omega)$, $n(\Omega)$, and also between the derivatives $\Phi'(\Omega)$ and $\Psi'(\Omega)$ according to conditions (4.4).

Consequently, the functions (4.3) express the solution of the equations of motion in potentials (4.2), if the argument Ω with coefficients, l, m, n, which are subject to Eq. (4.5), is determined by Eq. (2.5), while the functions (4.3) themselves satisfy conditions (4.4).

Assuming $l(\Omega) = 1$, $m(\Omega) = \theta$, $n(\Omega) = \lambda$ and $K(\Omega) = -\Omega$ in Eq. (2.5), we obtain the simplest solutions of the equations of motion (4.2). In this case the roots λ_k of Eq. (4.5) have the values (2.9).

Introducing the functions $\Phi_k(\Omega_k)$ and $\Psi_k(\Omega_k)$, corresponding to the roots λ_k, we conclude that the solutions of Eqs (4.2), which describe plane waves, are expressed by the functions

$$\varphi_k=\Phi_k(\Omega_k),\quad \psi_k=\Psi_k(\Omega_k)\qquad \Omega_k=t+\theta x+\lambda_k y,\quad k=1,2 \qquad (4.6)$$

Conditions (4.4) take the form

$$-\theta\left(p_k+c\lambda_k^2\right)\Phi_k'(\Omega_k)+\lambda_k\left(p_k-c\theta^2\right)\Psi_k'(\Omega_k)=0$$
$$\lambda_k\left(r_k+c\theta^2\right)\Phi_k'(\Omega_k)+\theta\left(r_k-c\lambda_k^2\right)\Psi_k'(\Omega_k)=0 \qquad (4.7)$$

and establish a relation between the functions (4.6)

Previously [2], the first condition of (4.7) was used to construct a solution in potentials. In this case the relation between functions (4.6) is expressed by the condition

$$-\frac{\Phi_k(\Omega_k)}{\lambda_k\left(p_k-c\theta^2\right)}=\frac{\Psi_k(\Omega_k)}{\theta\left(p_k+c\lambda_k^2\right)}=f_k(\Omega_k) \qquad (4.8)$$

Solution (4.6) takes the form

$$\varphi_k=-\lambda_k\left(p_k-c\theta^2\right)f_k(\Omega_k),\quad \psi_k=\theta\left(p_k+c\lambda_k^2\right)f_k(\Omega_k) \qquad (4.9)$$

Taking relations (4.1) into account, we obtain a solution of Eqs (4.2) in displacement

$$u_k=c\theta\lambda_k\left(\theta^2+\lambda_k^2\right)f_k'(\Omega_k),\quad \upsilon_k=-p_k\left(\theta^2+\lambda_k^2\right)f_k'(\Omega_k) \qquad (4.10)$$

Similarly, the second condition of (4.7) leads to a solution of Eqs (4.2)

$$\varphi_k=\theta\left(r_k-c\lambda_k^2\right)f_k(\Omega_k),\quad \psi_k=\lambda_k\left(r_k+c\theta^2\right)f_k(\Omega_k) \qquad (4.11)$$

The solution in displacements has the form

$$u_k=r_k\left(\theta^2+\lambda_k^2\right)f_k'(\Omega_k),\quad \upsilon_k=-c\theta\lambda_k\left(\theta^2+\lambda_k^2\right)f_k'(\Omega_k) \qquad (4.12)$$

Solutions (4.10) and (4.12), apart from the factor $(\theta^2 + \lambda_k^2) \neq 0$, agree with solutions (2.11) and (2.12) of equations of motions (2.1), obtained using one of the two conditions (2.10). Consequently, solutions (4.9)–(4.12) of the equations of motion in potentials (4.2), obtained using one of two conditions (4.7), do not agree with the physical meaning of the problem and with solutions (1.1) for an isotropic medium. It follows from relations (4.10), (4.12) and (2.18) that the solutions of the equations of motion in potentials (4.2) can be obtained as the sum of solutions (4.9) and (4.11).

These solutions can be obtained using the generalized condition obtained by summing conditions (4.7), i.e.

$$\left[\lambda_k\left(r_k+c\theta^2\right)+\theta\left(p_k+c\lambda_k^2\right)\right]\Phi_k'(\Omega_k)+\left[\lambda_k\left(p_k-c\theta^2\right)-\theta\left(r_k-c\lambda_k^2\right)\right]\Psi_k'(\Omega_k)=0 \qquad (4.13)$$

According to condition (4.13), the solutions of Eqs (4.2), which describe plane waves, take the form

$$\begin{aligned}\varphi_k &= -\left[\lambda_k\left(p_k - c\theta^2\right) - \theta\left(r_k - c\lambda_k^2\right)\right] f_k(\Omega_k)\\ \psi_k &= \left[\theta\left(p_k + c\lambda_k^2\right) + \lambda_k\left(r_k + c\theta^2\right)\right] f_k(\Omega_k)\\ \Omega_k &= t + \theta x + \lambda_k y, \quad k = 1,2\end{aligned} \tag{4.14}$$

where f_k are the branches of an arbitrary continuous triply differentiable function f, if the coefficients of the variables are real. If some of these coefficients in some region x, y, t are complex quantities, then f in this region will be an analytical function. Solutions (4.14), like solutions (2.18), are uniquely defined on Riemann surfaces, the form of which is shown in Figs 1 and 2.

We will consider the construction of complex solutions of a general type for the equations of motion in potentials (4.2), taking $l(\Omega) = 1$, $n(\Omega) = \lambda$ in Eq. (2.5) and taking the quantity θ, defined by the expression $m(\Omega) = \theta$, as the new variable. In this case λ and $K(\Omega)$ will be functions of the variable θ, while Eq. (2.7) will have roots λ_1 and λ_2, defined by (2.9).

Solutions (4.3) of the equations of motion (4.2) can be expressed by the functions

$$\varphi_k = \Phi_k(\theta_k), \quad \psi_k = \Psi_k(\theta_k) \tag{4.15}$$

while Eq. (2.5) has the form

$$\delta_k = t + \theta_k x + \lambda_k y + K_k(\theta_k) = 0 \tag{4.16}$$

where K_k are the branches of a certain analytical function K.

The conditions which establish the relations between functions (4.15), according to relations (4.4), are expressed by equations, by summing which, we obtain a generalized condition, which differs from condition (4.13) by replacing θ and Ω_k by θ_k.

According to this generalized condition, the general real solution of Eqs (4.2) is given by the expressions

$$\begin{aligned}\varphi(x,y,t) &= -\sum_{k=1}^{2} \operatorname{Re}\left\{\int^{\theta_k}\left[\lambda_k\left(p_k - c\varsigma^2\right) - \varsigma\left(r_k - c\lambda_k^2\right)\right] w_k(\varsigma)\,d\varsigma\right\}\\ \psi(x,y,t) &= \sum_{k=1}^{2} \operatorname{Re}\left\{\int^{\theta_k}\left[\lambda_k\left(r_k + c\varsigma^2\right) + \varsigma\left(p_k + c\lambda_k^2\right)\right] w_k(\varsigma)\,d\varsigma\right\}\end{aligned} \tag{4.17}$$

The correspondence between the points of the Riemann surface (Figs 1 and 2) and points in the xy plane is expressed by Eq. (4.16).

The zero-dimensional homogeneous solutions of the equations of motion (4.2), which describe elastic vibrations induced by a point source of the instantaneous impulse type at the origin of coordinates, can be obtained as a special case of the general solutions if we assume that $K_k(\theta_k) = 0$ in Eq. (4.16). in this case Eq. (4.16), which expresses the correspondence between points of the Riemann surface and the xy plane, takes the form

$$1 + \theta_k \xi + \lambda_k \eta = 0 \quad (\xi = x/t, \ \eta = y/t) \tag{4.18}$$

The homogeneous solutions of equations of motion (4.2) are expressed by functions (4.17).

The functions w_1 and w_2 are branches of an arbitrary analytical function w, uniquely defined on the Riemann surface shown in Fig. 1 when condition (2.22) is satisfied, and Fig. 2 when condition (2.27) is satisfied. In order that solution (4.17) should express elastic vibrations, induced by a point source in an unbounded medium, the function w must be chosen so that the real parts of the functions w_1 and w_2 vanish on the following edges of the cuts of the θ_1 and θ_2 planes of the real definition of the functions λ_1 and λ_2: when condition (2.22) is satisfied-on the edges of the cuts $(-a^{-1/2}, +a^{-1/2})$ of the θ_1 plane and $(-a^{-1/2}, +d^{-1/2})$ of the θ_2 plane (Fig. 1); when condition (2.27) is satisfied – on the edges of the cuts $(-a^{-1/2}, +a^{-1/2})$ and $(\pm d^{-1/2}, \pm\theta_1^\circ)$ of the θ_1 plane and $(-\theta_1^\circ, +\theta_1^\circ)$ of the θ_2 plane (Fig. 2). The wave fronts in the ζ, η plane correspond to the edges of these cuts [12, 13], which are expressed as the envelopes of straight lines (4.18) for real values of θ_k and λ_k.

5. CONCLUSION

Note that when using the Smirnov and Sobolev method to solve the equations of motion in displacements (2.1) and potentials (4.2), we arrive at conditions (2.6) and (4.4) for the components of the displacement vectors and for the scalar and vector potentials.

For plane waves, conditions (2.6) and (4.4) take the form (2.10) and (4.7). We have established that solutions (2.11) and (2.12) of the equations of motion in displacements, obtained, as previously in [1], on the basis of one of the two conditions (2.10), determine, in the directions of the axes of elastic symmetry of the medium, the propagation of only one type of wave – quasi-longitudinal or quasi-transverse, which does not agree with the physical meaning of the problem and with the solutions of the wave equations of isotropic media. We have a similar pattern for solutions (4.9) and (4.11) of the equations of motion in potentials, obtained, as previously in [2], from one of the two conditions (4.7).

We have established that the solutions of the equations of motion in displacements and potentials, obtained as the sum of solutions (2.11), (2.12) and (4.9), (4.11), agree with the physical meaning of the problem and with the solutions of the wave equations of isotropic media. These solutions in displacements and potentials can be obtained directly from the generalized conditions (2.17) and (4.13), which follow from conditions (2.10) and (4.7).

We have proposed a new method of constructing the solutions of the equations of motion (2.1) and (4.2) using generalized conditions for the components of the displacement vectors and for the scalar and vector potentials, obtained from conditions (2.6) and (4.4). We have obtained and investigated solutions of the equations of motion in displacements and potentials, which express plane waves and waves from point sources, and also complex solutions of a general type. For comparison with the results obtained in [1] we have solved the problem of the reflection of plane waves from a free boundary of an anisotropic half-space.

Despite some complexity of the solutions, their expression in terms of reverse apparent velocities in the directions of the axes of elastic symmetry and their unique determination on the Riemann surfaces (Figs 1 and 2) enables them to be used to solve a number of specific problems and to carry out analytical investigations of fairly complex wave processes in anisotropic media.

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